Gravity Systems

where

- $M_x$ is the yield moment along the $x$-axis
- $M_y$ is the yield moment along the $y$-axis
- $M_z$ is the yield moment along the $z$-axis

Observe that for the special case of a slab with the same reinforcement in each direction

$$M_x = M_y = M$$

resulting in

$$M_z = M (\cos^2 \alpha + \sin^2 \alpha) = M$$

This in fact, was the case in the previous example.

2.3.4.4 Limitations of Yield-Line Method

In using this method, it must be borne in mind that the analysis is predicted on the assumption that adequate rotation capacity exists at the yield lines. If this is not the case, it is likely that the required rotation will exceed the available rotation capacity leading to premature failure. However, in general, building slabs are lightly reinforced and will have adequate rotation capacity to attain loads predicted by yield-line analysis. It should also be kept in mind that the yield-line analysis is based only on the moment capacity of the slab. It is presumed that earlier failures will not occur due to bond, shear, or other causes. Although yield-line analysis gives no indication on deflections, stresses, or severity of cracking under service loads, it is an excellent tool for justifying moment capacity of existing slabs that are otherwise satisfactory under service load conditions.

2.3.5 Deep Beams

Deep beams are members loaded on one face and supported on the opposite face so that compression struts can develop between the loads and line supports, and have either of the following:

1. The clear span, $L$, is less than or equal to four times the overall depth of beam
2. Concentrated loads occur very near the supports (within twice the member depth from the face of the support)

There are two choices for designing deep beams: (1) by taking into account nonlinear distribution of strain or (2) by using the strut-and-tie models as given in ACI 318.08 Appendix A.

Some of the design requirements are

1. The nominal shear strength, $V_n$, shall not exceed $10 \sqrt{f_y} b_x d$.
2. The area of shear reinforcement, $A_{s_y}$, shall not be less than $0.0025 b_y s$. The spacing, $s$, shall not exceed $d/5$ or 12 in.
3. The area of shear reinforcement, $A_{s_{y1}}$ parallel to the flexural tension reinforcement shall not be less than $0.0015 b_y s_2$. The spacing $s_2$ shall not exceed $d/5$ or 12 in.
   In lieu of the aforementioned minimum shear reinforcement, we are permitted to provide reinforcement that satisfies the requirement of strut-and-tie models.
4. The minimum area of flexural tension reinforcement, $A_{s_{y1}}$, shall not be less than

   $$A_{s_{y1}} = 3 \sqrt{\frac{f_y}{f_y}} b_y d \quad \text{or} \quad 200 \frac{b_y d}{f_y}$$

5. Where the overall depth, $h$, exceeds 36 in., longitudinal skin reinforcement is required to control cracking in the web near the tension zone.
The ACI 318-08 like its predecessors does not contain detailed requirements for designing deep beams for flexure except that nonlinearity of strain distribution and lateral buckling is to be considered.

As mentioned previously, when the span-to-depth ratio of a beam is less than or equal to four, it is customary to define these beams as deep. The traditional principles of stress analysis using the engineers bending theory, ETB are neither suitable nor adequate.

The stresses in homogeneous deep beams before cracking can be determined using sophisticated analysis such as a finite element solution. These analyses indicate that the smaller the span/depth ratio, the more pronounced the deviation of the stress from the ETB. As an example, Figure 2.34 shows the distribution of bending stresses at midspan of simply supported beams having different span-depth \((l/h)\) ratios, when carrying a uniformly distributed load. It is noted from Figure 2.34 that for \(l/h = 1\), the tensile stresses are more than twice the intensity obtained from the simple building theory.

Considering again the square beam \((l/h = 1.0)\), two observations may be made from Figure 2.34. First, the tension zone at the bottom of the beam is relatively small, approximately equal to 0.25 \(\ell\), suggesting that the tension reinforcement should be placed in this area. Second, the tensile force, and hence the reinforcement that is of primary interest, could be computed by using the internal

![Diagram](image-url)

**FIGURE 2.34** Stress distribution in deep beams: (a) \(l/h = 4\), (b) \(l/h = 2\), (c) \(l/h = 1\), and (d) \(l/h < 1.0\).
lever arm, \(j d = 0.62h\). It is interesting that this is approximately the same for all beams; that is, it is not affected greatly by the span/depth ratio, \(l/h\).

2.3.6 **Strut-and-Tie Method**

The strut-and-tie method is a simple and intuitive method based on static equilibrium.

The method is typically applied to structural elements in which the assumption of simple bending theory do not strictly apply. One such assumption is that in a flexural member such as a beam, plane sections before bending remain plane after bending. This assumption is valid at all cross sections of the beam except at the immediate proximity to applied loads and reactions. An example of a member in which plane sections do not remain plain is a beam with clear span, \(l_o\), equal to or less than four times the over all member depth. Therefore, instead of using the simple bending theory, we use a more appropriate approach such as a truss analogy to define a load path. The analysis begins by assuming an internal load path, consisting of appropriate struts and ties within the member being designed, and then designing the elements for the resulting forces.

Structural elements in a typical load path consist of a truss model that has

1. Inclined and vertical compressive struts
2. Longitudinal tension members also called ties
3. Node regions at all joints of chords, struts, and ties

The sizes of the members and joint regions in the truss model are chosen so that the computed demand forces in the struts, ties, and the nodes due to factored loads will not exceed respective design capacity. It should be kept in mind that for the mobilization of the tensile reinforcement, the tension ties shall be effectively anchored to transfer the required tension to the truss node regions.

The paramount requirement for the safety of a design using the strut-and-ties method is that the member must have adequate ductility to enable redistribution of actions to the designated load path.

It is worth noting that for any given condition, more than one truss configuration can be selected to resist the applied loading. While one truss configuration may be more efficient than another from the design standpoint, it is sufficient to demonstrate that the chosen truss can sustain the load, and has adequate ductility to mobilize it.

Before further discussing the truss-and-tie model, it is perhaps instructive to briefly dwell on the so-called-Saint-Venant’s Principle on which the analysis is based. In discussing the elementary theory of simple bending, commonly referred to as engineers theory of building, ETB, French elastician Saint-Venant (1797–1886) formulates the principle which now carries his name. He states that the stress distribution given by the elementary bending theory is correct only when the external forces are applied to the member in the same manner as the bending stresses are distributed over intermediate cross sections. He further states the solutions obtained from the ETB will be accurate enough in most cross sections except at the immediate vicinity of the applied forces and reactions.

To get an insight into this principle, consider Figure 2.35 that shows one of Saint-Venant’s examples. The two equal and opposite forces exerted by the clamp produce only a local deformation. Hence stresses are produced only in the vicinity of clip, and at a distance sufficiently far away from the clip, the tube is practically unaffected. The critical distance over which the local effects occur is typically taken equal to one to two times the characteristic dimension of the member.

The strut-and-tie method of analysis for the design of deep beams given in ACI 318-08, Appendix A is based on the Saint-Venant’s principle described above. The pinched region of tube shown in Figure 2.35 is similar to the Discontinuity region described in the ACI 318. This region is assumed to extend no more than one-member depth from the point of application of loads and reactions. The Bending regions, \(B\) referenced in the ACI on the other hand are those that are sufficiently far away from the regions of discontinuity where the ETB may be applied without significant errors.
FIGURE 2.35  Example of Saint-Venant’s principle: pinched region is similar to the discontinuity, D, region (ACI 318-08, Appendix A). Regions away from D are bending, B, regions.

FIGURE 2.36  Strut-and-tie terminology: (a) strut-and-tie model, (b) C-C-C node resisting three compressive forces, (c) and (d) C-C-T nodes resisting two compressive, and one tensile force.

Certain terms unique to strut-and-tie models are defined in the ACI 318. The first term nodal zone, shown in Figure 2.36, describes the volume of concrete around a node that is assumed to transfer the strut-and-tie axial forces through the nodes. The second term strut is a compression member while the third term tie refers to a member carrying tension.

The design methodology is quite similar to what we typically use in the ultimate design of concrete members: we compare the calculated internal forces, $F_u$, in the struts, ties, and nodal zones to the usable capacity $\phi F_u$ and declare the design to be satisfactory if

$$\phi F_u \geq F_u$$
The design of deep beams using strut-and-tie models is essentially by a trial-and-error method. Hence it is well suited for an interactive spread sheet application.

The design steps are

1. Assume the center of gravity of the lower tension chord AC is located at a certain distance above the bottom of beam. A good value to use is 0.05D where D is the beam depth.
2. Similarly assume the center of gravity of the top node B at a distance of 0.05D below the top of the beam.
3. The above assumptions permit us to define the geometry and hence the flow of forces, tension in the bottom chord AC, and compression in the struts AB and BC. The resulting forces denoted by $F_u$ are the factored forces in the strut, tie, or on the face of node.
4. Compute the nominal strength, $F_{n}$, of the strut, tie, or nodal zone using the following equations:

   \[
   \text{Struts: } f_{ce} = 0.85 \beta_e f'_{c}
   \]

   where
   - $\beta_e = 1.0$ for prismatic struts
   - $\beta_e = 0.75$ for bottle-shaped struts with reinforcements
   - $\beta_e = 0.60 \lambda$ for bottle-shaped struts without reinforcement. $\lambda$ is the reduction factor for lightweight concrete (See ACI Section 8.6.1)

   \[
   \text{Ties: } F_u = A_n f_y \text{ (for non-prestressed ties)}
   \]

Note that tension reinforcement shall be anchored by straight bar development, standard hooks, or mechanical devices.

\[
\text{Nodal zones: } F_{in} = f_{ce} A_{nz}
\]

where
- $F_{in}$ is the nominal compression of a nodal zone
- $A_{nz}$ is the area of the nodal face on which $F_u$ acts
- $f_{ce}$ is the effective compressive stress on a face of a nodal zone = $0.85 \beta_e f'_{c}$

where
- $\beta_e = 1.0$ for a C-C-C node
- $\beta_e = 0.8$ for a C-C-T node
- $\beta_e = 0.60$ for C-C-T node

**Design Example**

**Given:** A single span deep beam, 31 ft 10 in. long and 15 ft deep with a carried column at the center of span. The ultimate load including the weight of beam is 6867 kip. See Figure 2.37.

**Required:** Design of transfer beam using strut-and-tie method.

**Solution:**

Step 1 Select a strut-and-tie model consisting of two struts, one tie, and three nodal zones. Compute the reactions at A and B after adding the ultimate dead load of the beam to the column load. In our case, total load $P_u = 6867$ kips as given in the statement of the problem. Therefore,

\[
R_A = R_B = \frac{6867}{2} = 3433.5 \text{ kip}
\]

FIGURE 2.38  Strut-and-tie model: (a) C-C-C node and (b) C-C-T node.

Step 2  For the first trial, assume the height $h_b$ of the bottom nodes A and B, and the height $h_t$ of the top node C, to be the same, at $0.05D = 0.05 \times 15 \text{ ft} = 0.75 \text{ ft} = 9 \text{ in}$.

Step 3  Effective compression strength, $\phi f_{ce}$, for the nodal zones.

Using the ACI 318 nomenclature, nodes B and C are classified as C-C-T nodes meaning that the nodes resist two compressive forces, and one tensile force. The node at the top, below the carried column, is referred to as a C-C-C node meaning all forces resisted by this node are compressive. (see Figure 2.38).

For the C-C-T nodes A and C, the effective compressive stress, $f_{ce}$, from ACI Section A.5.2 is

$$f_{ce} = 0.85\beta_{bo} f'_c$$

In this equation, $\beta_{bo}$ is a penalty factor that measures how well the tension member (if any) is anchored at a node. If the node is a C-C-C node (meaning no ties), there is no penalty. Hence $\beta_{bo} = 1.0$. On the other extreme, if the node is a C-T-T node, $\beta_{bo} = 0.6$ and if it is a C-C-T node, $\beta_{bo}$ is the average of the two $= 1.0 + 0.6/2 = 0.8$.

Referring to nodes A and C,

$$f_{ce} = 0.85 \times 0.8 \times f'_c = 0.68 f'_c = 0.68 \times 10,000 \text{ psi} = 6.8 \text{ ksi}$$

For node B  $f_{ce} = 0.85 \times 1.0 \times f'_c = 8500 \text{ psi} = 8.5 \text{ ksi}$
Step 4  Compute effective compression strengths for struts AB and BC.

From ACI 318-08 Section A.3.2:

\[ f_{ce} = 0.85 \beta f_c' \]

For a strut of uniform section, \( \beta = 1.0 \).

Therefore, \( f_{ce} = 0.85 \times 1.0 \times 10,000 = 8500 \text{ psi} = 8.5 \text{ ksi} \)

Schematic results obtained from a spreadsheet for the transfer beam shown in Figure 2.37 are summarized in Figures 2.39 through 2.42. A practical reinforcement layout for the transfer girder and adjacent beams is shown in Figure 2.43.

**FIGURE 2.39**  Member stress limits and effective widths.

**FIGURE 2.40**  Factored forces and design strengths.

**FIGURE 2.41**  Stress ratios.
### Struts

<table>
<thead>
<tr>
<th>Strut ID</th>
<th>Demand $F_x$ (k)</th>
<th>$\beta_\alpha$</th>
<th>$\phi$</th>
<th>Effective Width (in.)</th>
<th>Effective Thickness Scale Factor (in.)</th>
<th>Capacity $\phi F_m$ (k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>-5461.62</td>
<td>0.750</td>
<td>0.750</td>
<td>4781.0</td>
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<td>30.00</td>
<td>1.000</td>
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<td>1.000</td>
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### Ties

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<thead>
<tr>
<th>Tie ID</th>
<th>$F_x$ (k)</th>
<th>Required $A_t$ (in.²)</th>
<th>Provided $A_t$ (in.²)</th>
<th>$\phi$</th>
<th>$\phi F_m$ (k)</th>
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### Nodes

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<th>Node Face</th>
<th>Demand $F_x$ (k)</th>
<th>$\beta_\alpha$</th>
<th>$\phi$</th>
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<th>Effective Thickness Scale Factor (in.)</th>
<th>Capacity $\phi F_m$ (k)</th>
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<td>30.00</td>
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</table>

**FIGURE 2.42** Design Summary. *Note:* Struts E4, E5, and E6 are transfer column and supporting columns. Nodes N4, N5, and N6 are at the face of transfer column and supporting columns.